Simplified Settings for Discrete Logarithms in Small Characteristic Finite Fields

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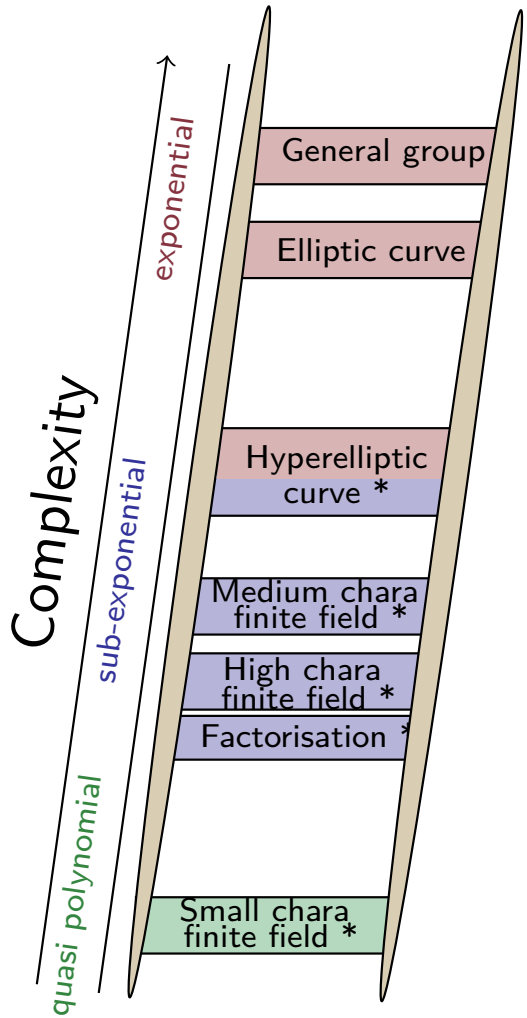

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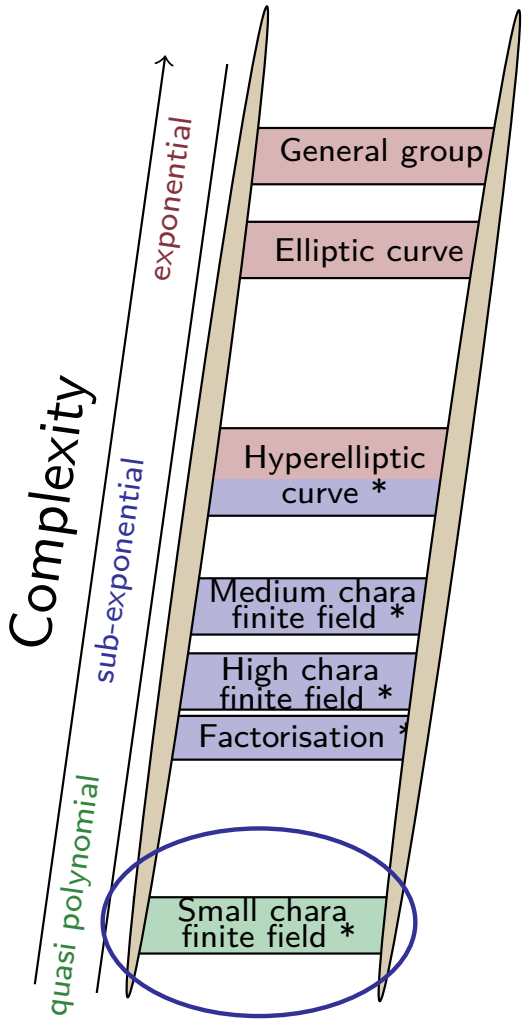

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  - Generic algorithms (Pollard's Rho, Pohlig-Hellman...)
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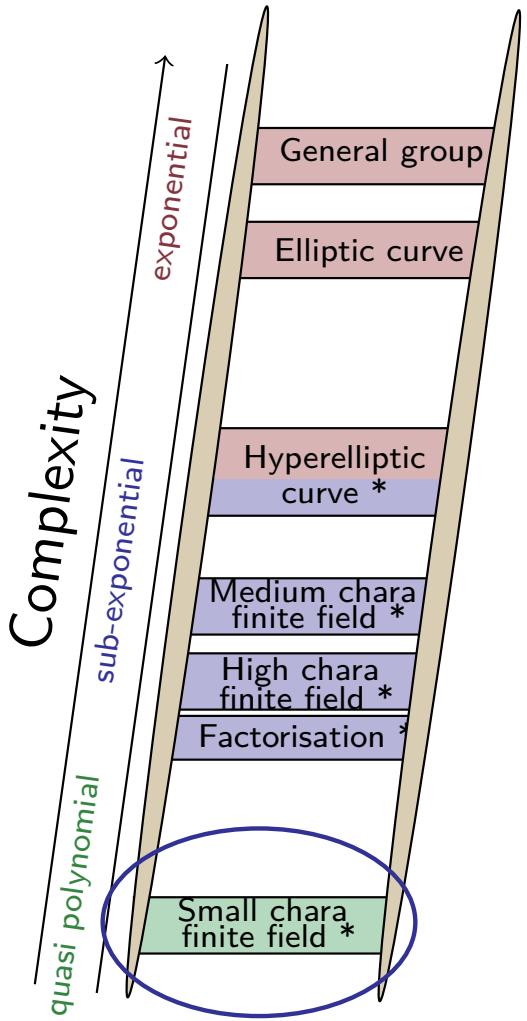

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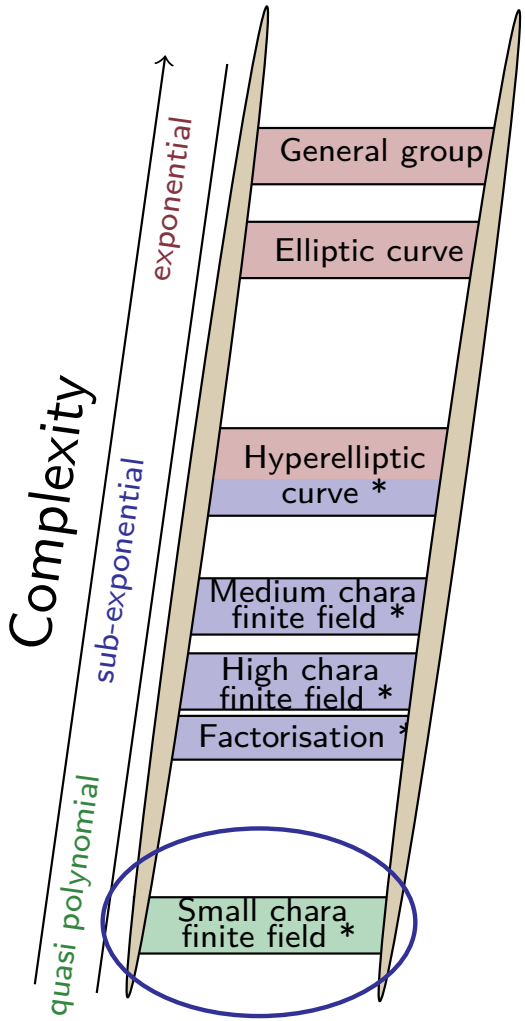

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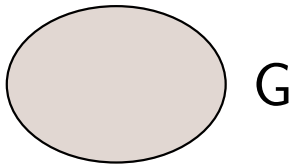

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   $\rightarrow$ Create a lot of sparse multiplicative relations between some (small) specific elements = the factor base

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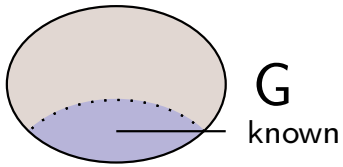

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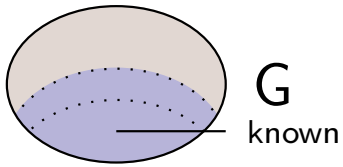

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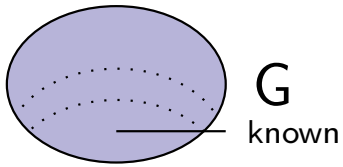

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4. Individual Logarithm Phase
   $\rightarrow$ Recover the Discrete Log of an arbitrary element

Complexities

- Asymptotic Complexities:

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- But in practice:
  Linear algebra and extension phases dominate
- What do we do ?
  Simplified description of algorithms + additional ideas
  $\Rightarrow$ Improve the complexity of the polynomial phases.

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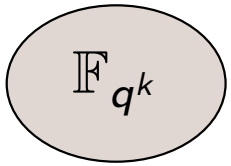

- How ? Represent $\mathbb{F}_{q^k} \simeq \mathbb{F}_q[X]/(I(X))$ where $I(X)$ is an irreducible polynomial of degree k such that:

$$I(X) | h_1(X)X^q - h_0(X)$$

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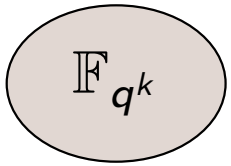

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$$\prod_{\alpha \in \mathbb{F}_q} (X - \alpha) = X^q - X \qquad \text{and} \qquad \underbrace{X^q = \frac{h_0(X)}{h_1(X)}}_{\text{Frobenius Representation}}$$

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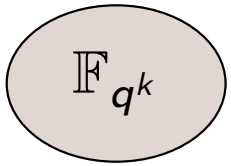

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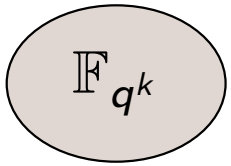

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h_0 : deg 1 polynomial
h_1 : deg 2 polynomial

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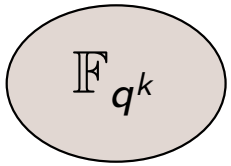

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$$\left.\begin{array}{ll} h_0 : & \text{deg 1 polynomial} \\ h_1 : & \text{deg 2 polynomial} \end{array}\right. \quad \text{or} \quad \left.\begin{array}{ll} h_0 : & \text{deg 2 polynomial} \\ h_1 : & \text{deg 1 polynomial} \end{array}\right\} \text{useful variant}$$

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Main idea : start from $\prod_{\alpha \in \mathbb{F}_q} (X - \alpha) = X^q - X \quad (**)$.

Let A and B be 2 small polynomials in $\mathbb{F}_q[X]$ (i.e. of degree $\leqslant D$).

$$B(X) \prod_{\alpha \in \mathbb{F}_q} (A(X) - \alpha B(X)) \underset{\text{thanks to } (**)}{=} A(X)^q B(X) - A(X) B(X)^q$$

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We finally get:

$$\underbrace{h_1(X)^D B(X) \prod_{\alpha \in \mathbb{F}_q} (A(X) - \alpha B(X))}_{\text{Product of small polynomials !!}} = [A,B]_D(X)$$

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- But... it's not possible to $\searrow D$, isn't it? Previous constraints:
  1. Need to generate enough good equations = equations where $[A, B]_2$ splits in terms of degree $\leqslant 2$. Pb: the probability $\mathcal{P}$ to have good equations is too small w.r.t the number of equations required (need $\mathcal{P} > 1/2$).
  2. Need to be able to descent large polynomials to degree 2 ones.

A Reduced Factor Base: Systematic factors of $[A, B]_D$

- Our goal, solving pb 1: i.e. improve the probability $\mathcal{P}$.
- How ? $[A, B]_2$ is a degree 6 polynomial. The prob that it factors into degree 2 polynomials is too low.

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  Yet, $[A, B]_D$ has a systematic factor of degree 3 ! Namely $X\, h_1(X) - h_0(X)$.
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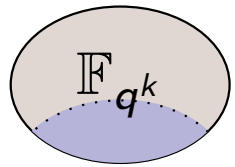

$\Rightarrow$ Linear Algebra permits to recover the DLogs of the factor base in $O((\underbrace{\#\text{ factor base}}_{q^2})^2(\underbrace{\#\text{ of entries}}_{q})) \approx O(q^5)$ operations.

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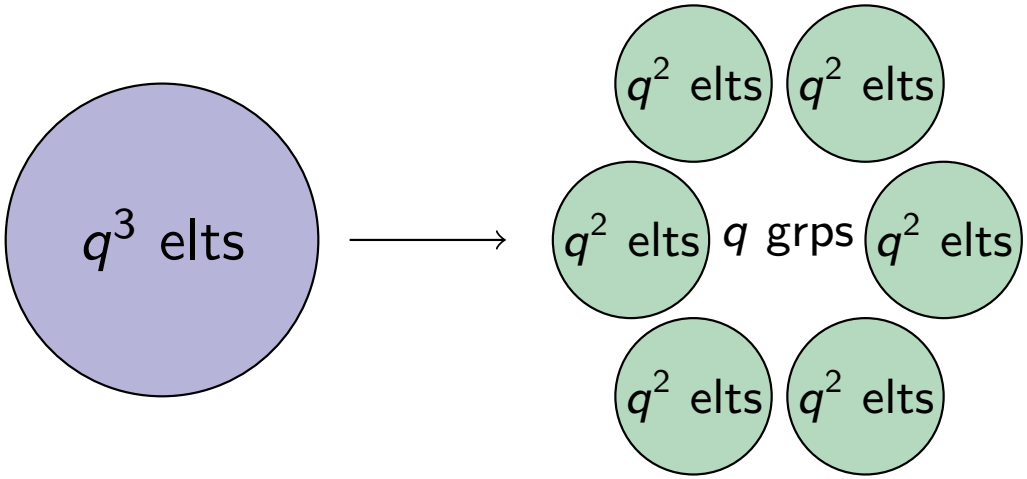

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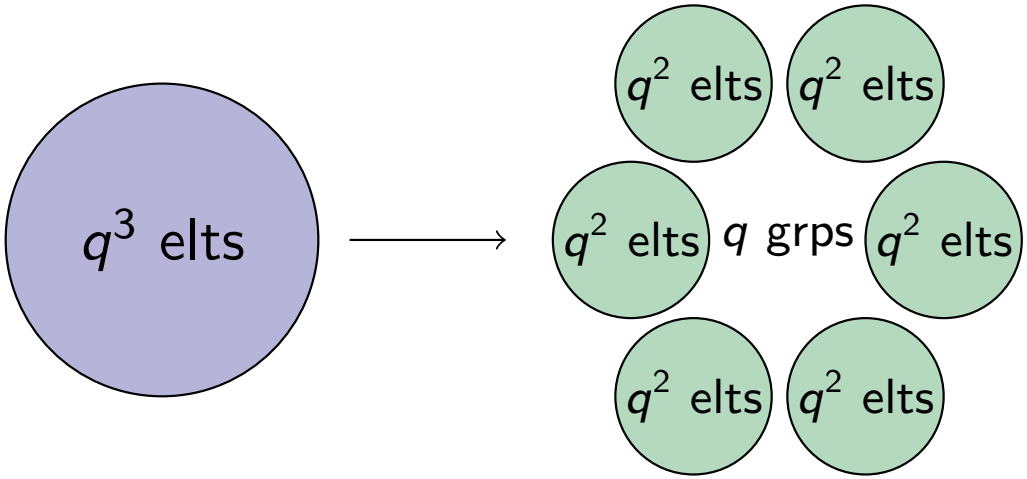

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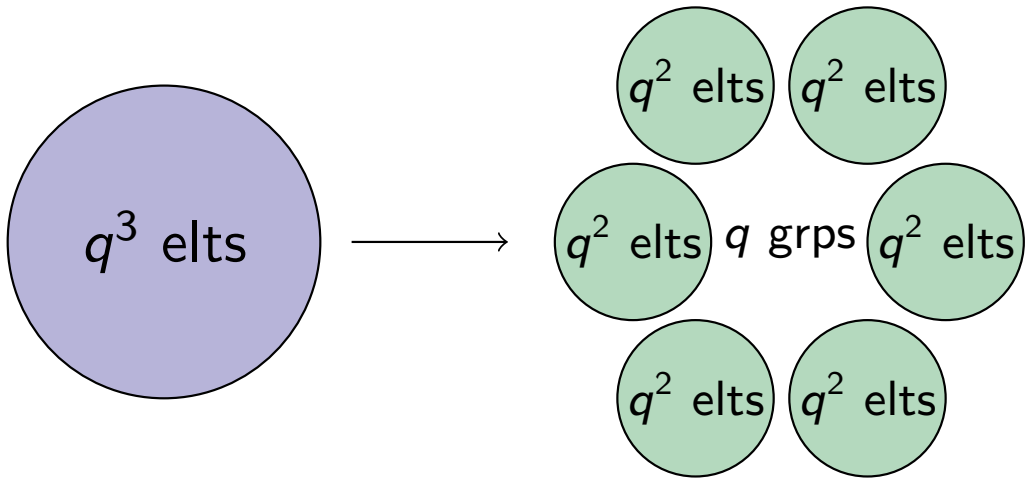

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2. Given q^2, generate equations involving only poly in q^2 and degree 1 and 2 polynomials (whose logs are already known).

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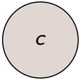

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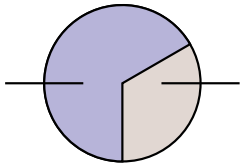

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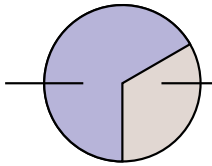

As for degree 2: set $A(X) = (X^3 + c) + \alpha X^2$ and $B(X) = (X^3 + c) + \beta X$ and create relations of the form:

$$\underbrace{h_1(X)^3 B(X) \prod_{\alpha \in \mathbb{F}_q} (A(X) - \alpha B(X))}_{\text{all belongs to } c \text{ !!}} = \underbrace{[A, B]_3(X)}_{\begin{array}{l}\text{deg 8 with these } A \text{ and } B \\ + \text{ deg 3 systematic factor} \\ + \text{ divisible by } X\end{array}}$$

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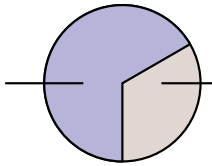

$\Rightarrow$ new unknowns

As for degree 2: set $A(X) = (X^3 + c) + \alpha X^2$ and $B(X) = (X^3 + c) + \beta X$ and create relations of the form:

$$\underbrace{h_1(X)^3 B(X) \prod_{\alpha \in \mathbb{F}_q} (A(X) - \alpha B(X))}_{\text{all belongs to } \mathcal{C} \text{ !!}} = \underbrace{[A, B]_3(X)}_{\substack{\text{deg 8 with these } A \text{ and } B \\ + \text{ deg 3 systematic factor} \\ + \text{ divisible by } X}}$$

Prob that $[A, B]_3$ factors into deg $\leqslant 3 \Rightarrow 41\%$. Enough !

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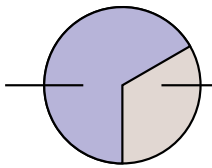

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- Complexity to recover the Dlogs of all degree 3 polynomials:

$$O(\underbrace{(\#\ \boxed{c})}_{q}\underbrace{(\#\ \text{factor base})^2}_{q^2}\underbrace{(\#\ \text{of entries})}_{q}) \approx O(q^6) \text{ ops.}$$

Extend the Factor Base to Degree 4

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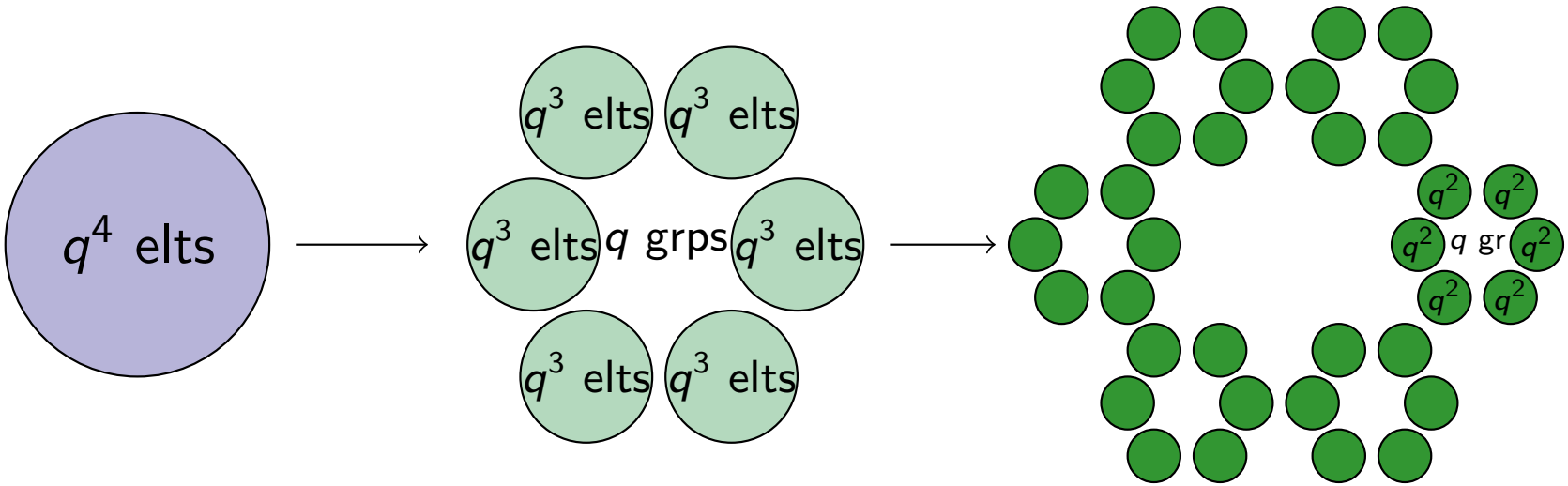

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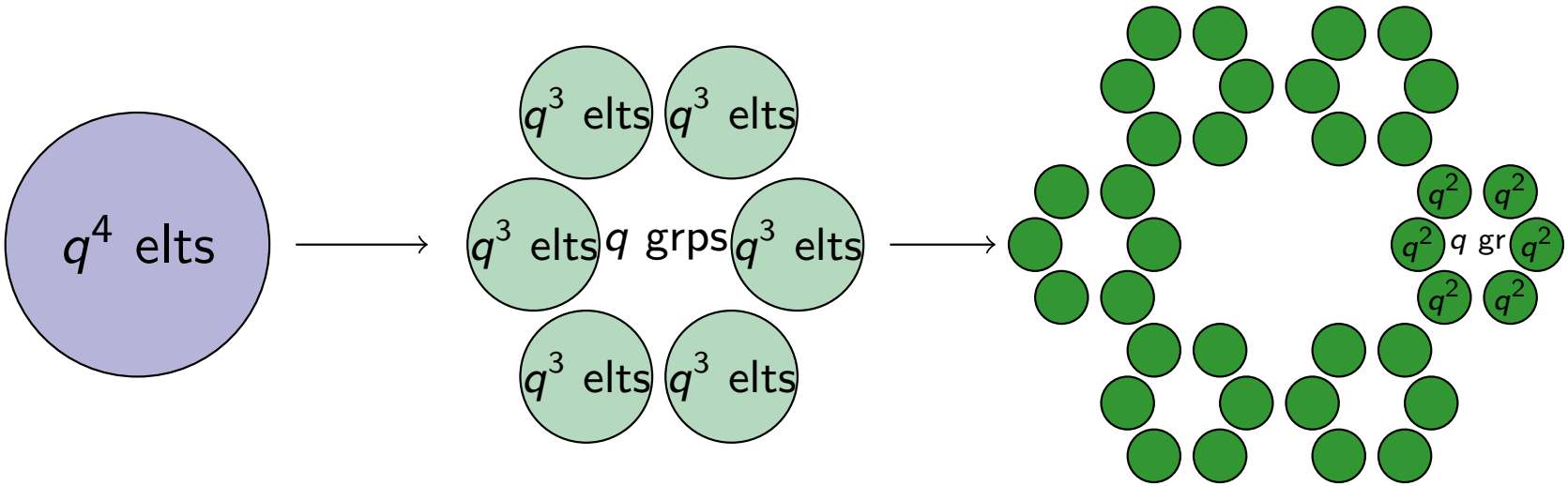

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2 poly belongs to the same q^3 if same constant coefficient.

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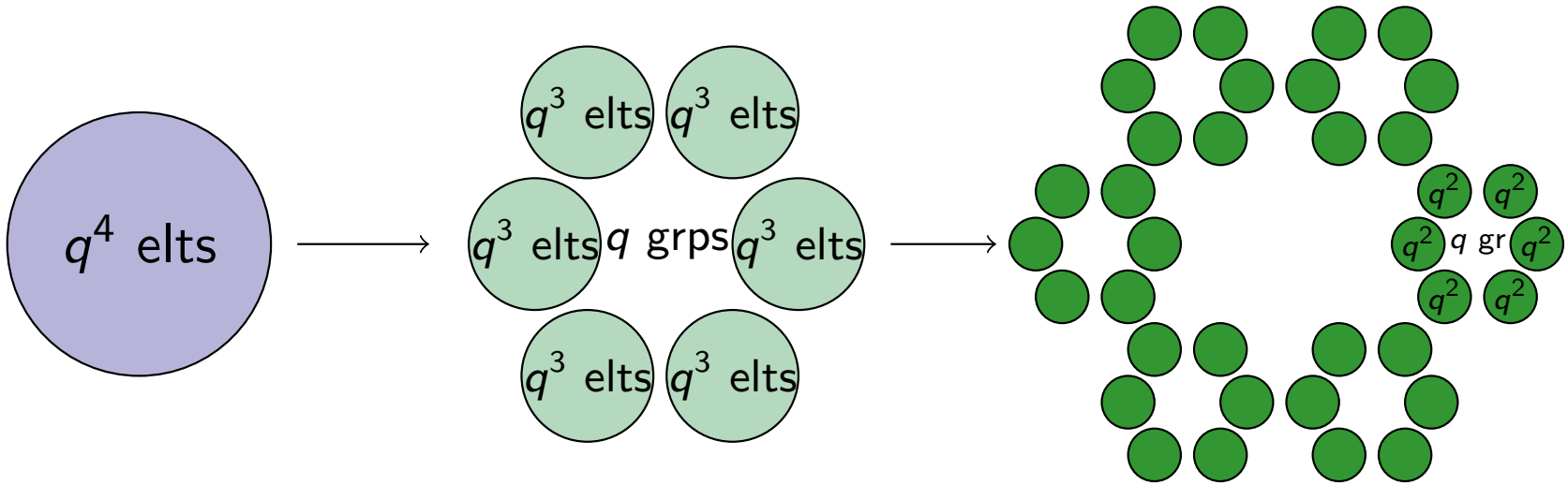

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2. Given q^2, generate equations involving only poly in it and degree 1, 2 and 3 polynomials.

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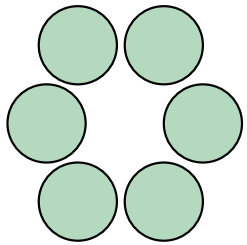

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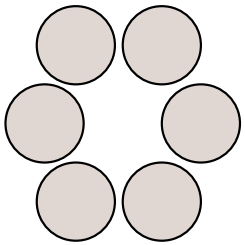

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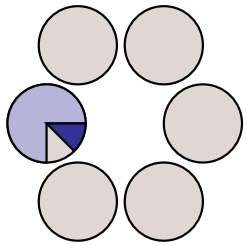

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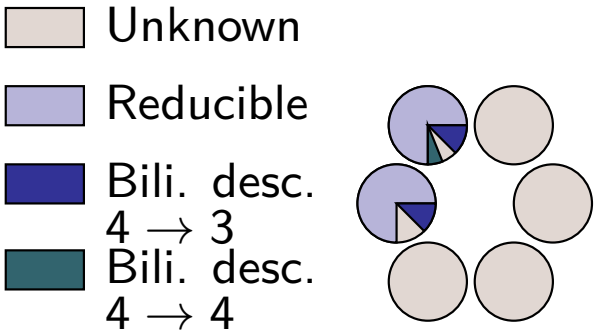

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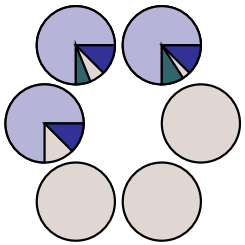

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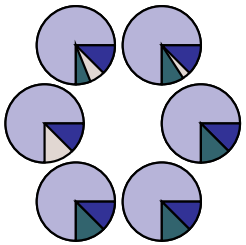

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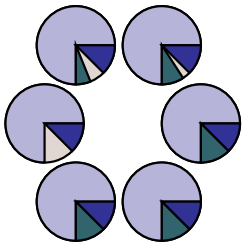

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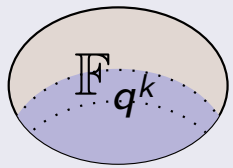
Main Result

Final asymptotic complexity of the three first phases:

$O(q^6)$ operations – to be compared with previous $O(q^7)$.



And in practice ?

- New record in characteristic 3 on $\mathbb{F}_{3^{5\cdot 479}}$, a finite field of cardinality a 3796-bit integer.
  - Not a special extension field such as Kummer extension !
  - Make use of the Dual Frobenius Representation combined with the useful variant (both not presented here).
- To be compared with previous record in characteristic 3 by Adj, Menezes, Oliveira and Rodriguez-Henriquez on a 1551-bit finite field.
- Time : 8600 CPU-hours $\approx$ 1 CPU-year

Thank you for your attention !